Paper

Statistical analysis related to impulse tests for self-restoring insulation

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For dielectric tests with impulse voltage, some statistical methods with the step-up procedure have been recently proposed. In the present paper, the methods are reconsidered from the viewpoint of statistical independence concerning disruptive discharges. As a result, modified methods with the step-up procedure or with the up-and-down procedure are proposed for self-restoring insulation, and it is shown that they have good properties in the maximum likelihood estimation for the mean and the standard deviation of a normal distribution.

Keywords: Maximum likelihood estimation; Normal distribution; Dielectric breakdown; Impulse testing; Up-and-down test; Step-up test

1. Introduction

We are concerned with statistical methods dealing with dielectric tests with impulse voltage. The conventional methods are given by International Electrotechnical Commission (IEC) (1) or IEEE (2). On the other hand, two different types of methods with the step-up procedure have been recently proposed in a series of the papers by Hirose (3)∼(8).

The two types of methods are the step-up method, which was originally proposed in the papers (3) (4), and the new step-up method (5)∼(8). In the methods, each result in voltage stress applications is supposed to be statistically independent under an underlying probability distribution, regardless of whether a disruptive discharge occurs or not. This assumption is the same as that in the up-and-down method (1) (2), which is for self-restoring insulation. In spite of that, these methods have been proposed for non-self-restoring insulation.

Self-restoring insulation like liquid or gaseous electrical insulation is the insulation that completely recovers its insulating properties after a disruptive discharge caused by a voltage stress application (1) (2). In addition, after it recovers, another disruptive discharge can be caused by the application of another voltage stress independent of the voltage stress when the previous disruptive discharge occurred. On the other hand, non-self-restoring insulation like solid electrical insulation is the insulation that loses its insulating properties, or does not recover them completely, after a disruptive discharge caused by a voltage stress application (1) (2). Thus, we can see that there is a big difference in influence by a voltage application between self-restoring insulation and non-restoring insulation.

Now, a question arises: "is the step-up method (not the step-up procedure itself) really appropriate for non-self-restoring insulation?" Our first aim is to consider this question. After the consideration, our second aim is to disclose merits and demerits of the step-up method as a method for self-restoring insulation.

In the papers (5) (9), Hirose has pointed out that with the recent improvement of voltage measuring instruments, we can observe a voltage at the moment when a disruptive discharge occurs. For this, he has proposed the new step-up method and the new up-and-down method and has shown the superiority of them to the step-up method and the up-and-down method, respectively. The new methods, however, have a technical defect. Although a disruptive discharge can typically occur after the peak of an impulse voltage, it is not dealt with appropriately in the methods. (We will see details later.) Thus, our third aim is to propose a modified step-up method and a modified up-and-down method, which inherit merits of the new step-up method and the new up-and-down method and avoid the technical defect in them.

Throughout the present paper we use the word "method" to mean a combination of a test procedure and the way of analysis of test results, such as the up-and-down method.

In Section 2 we will introduce the step-up, the up-and-down, the new step-up and the new up-and-down methods and give some remarks about them. In Section 3 the step-up method and the up-and-down method will be compared in the asymptotic or empirical errors. In Section 4 a modified step-up method and a modified up-and-down method will be proposed and investigated. In Section 5, conclusions will be given. An iterative formula to get estimates and a formula to calculate an observed information matrix will be given in Appendix.

2. Preliminaries

We introduce the test procedures and the likelihood functions of the step-up, the up-and-down, the new step-up and the new up-and-down methods, respectively, and state some remarks.

2.1 Step-up method The test procedure proceeds as follows (3)∼(8).

i) Decide the first voltage level U_1, a small amount

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$\triangle U$ and a maximum number m of voltage stress applications at a voltage level. Here, U_1 should be sufficiently low such that almost no disruptive discharge occurs at the level.

ii) If no disruptive discharge occurs when substantially equal voltage stresses are applied m times at the voltage level U_1, set the next voltage level U_2 at $U_1 + \triangle U$. Similarly increase the succeeding voltage levels U_3, U_4, ... until a disruptive discharge occurs.

iii) Perform ii) n times.

The likelihood function has been given as follows (3)~(8). For the jth disruptive discharge ($1 \leq j \leq n$), denote by δ_j and m_j an integer for which a disruptive discharge occurs at U_{δ_j} and the number of voltage stress applications at the level. If $F(U;\boldsymbol{\theta})$ is the discharge probability distribution function ($\boldsymbol{\theta}$ denotes a vector of parameters), the likelihood function becomes:

$$L = \prod_{j=1}^{n} \left\{ F(U_{\delta_j};\boldsymbol{\theta})[1 - F(U_{\delta_j};\boldsymbol{\theta})]^{m_j-1} \times \prod_{k=1}^{\delta_j-1} [1 - F(U_k;\boldsymbol{\theta})]^m \right\}. \cdots\cdots\cdots (1)$$

All the test results are expressed by U_1, $\triangle U$ and $\{(\delta_j, m_j)\}_{j=1}^n$ As seen in (1), each result (discharge or withstand) in voltage stress applications is dealt with statistically independently. This method has been proposed for non-self-restoring insulation (3)~(8).

Electrical engineers sometimes employ the sample mean and the sample standard deviation as estimates of the mean parameter and the standard deviation parameter for a normal distribution. This approach has been criticized in the papers (3) (5), in which the authors claim that the approach is incorrect and insist that the likelihood analysis via (1) is necessary (3) (5). However, in this paper, we argue that the method commonly employed by electrical engineers is in fact correct, for the following reasons.

In order to see the reasons, let us consider the following likelihood function:

$$L = \prod_{j=1}^{n} f(\tilde{U}_j;\boldsymbol{\theta}), \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (2)$$

where $\tilde{U}_j$ denotes the voltage level at which a disruptive discharge occurs and $f(\tilde{U};\boldsymbol{\theta})$ denotes the discharge probability density function. In (2), note that only voltage levels $\tilde{U}_j$'s at which disruptive discharges occur are dealt with statistically independently and the other voltage levels U_i's before disruptive discharges occur do not appear in (2). In other words, disruptive discharge voltages on test objects are expressed by a independent random variable $\tilde{U}$, and $\tilde{U}_j$ corresponds to the disruptive discharge voltage of the jth test object.

When a normal distribution is assumed, that is,

$$f(\tilde{U};\boldsymbol{\theta}) = \exp[-(\tilde{U}-\mu)^2/2\sigma^2]/\sqrt{2\pi}\sigma$$

in (2), the maximum likelihood (ML) estimates of the parameters μ and σ are the sample mean and the sample standard deviation. Hence, the difference between the step-up method and the engineers' estimation method is the assumption of statistical independence. In a non-self-restoring specimen like solid insulation, it is hard to accept that the voltage at which a disruptive discharge occurs can vary entirely independently of the specimen itself for every voltage stress application. The voltage could rather depend on the specimen itself. For non-self-restoring specimens, thus, the engineers' assumption is more natural than that in the step-up method.

In addition, it should be noted that, in the papers (3) (5), the superiority of the step-up method is indicated by using data simulated under the assumption of statistical independence in the step-up method. Under the assumption, the probability that a disruptive discharge does not occur before the k_0th voltage level, is

$$\prod_{k=1}^{k_0-1} [1 - F(U_k;\boldsymbol{\theta})]^m,$$

which goes to 0 as $k_0 \to \infty$. Thus, by noting that k_0 such that $U_{k_0} > u$ for a $u > U_1$ becomes larger as $\triangle U$ becomes smaller, we can see that the statement "the smaller the $\triangle U_1$, the smaller the distribution of $U_{\triangle_j}$" (5) comes from the assumption.

2.2 Up-and-down method The up-and-down 50% disruptive discharge voltage test is defined as follows (1, p. 91) (2, p. 118).

i) Decide the first voltage level U_1, a relatively small value $\triangle U$ and the total number N of voltage stress applications.

ii) When a voltage stress is applied at the voltage level U_1, and no disruptive discharge occurs, set the next voltage level U_2 at $U_1 + \triangle U$. If it occurs, set that at $U_1 - \triangle U$.

iii) Perform similar tests at the succeeding voltage levels U_2, U_3, ..., U_N.

The likelihood function for the test is given as follows (1, p. 99) (2, p. 122). Denote by d_i the number of discharges found in a voltage application at a voltage level U_i. Since $d_i = 1$ or 0, the number of withstands is given by $1 - d_i$ at U_i. Hence, the likelihood function L becomes:

$$L = \prod_{i=1}^{N} [F(U_i;\boldsymbol{\theta})]^{d_i} [1 - F(U_i;\boldsymbol{\theta})]^{1-d_i}. \cdots\cdots\cdots (3)$$

All the test results are expressed by U_1, $\triangle U$ and $\{d_i\}_{i=1}^N$. As seen in (3), each result d_i in voltage stress applications is dealt with statistically independently. Now, we can see that the assumption of statistical independence is the same as that in the step-up method and the difference between the up-and-down method and the step-up method is only the way of obtaining data. Hence, by remembering that the ML methods permit estimation of any statistical quantities once the value of $\boldsymbol{\theta}$ has been determined, we can say that the up-and-down method and the step-up method are competitors each other in obtaining good estimate of $\boldsymbol{\theta}$. Eventually, since the up-and-down method is well-known as a method for self-restoring insulation, we can see that the step-up method is a method for self-restoring specimens rather than for non-self-restoring specimens.

Up to this subsection, we have introduced the methods that deal with censored data only. That is, all the information they use is whether a disruptive discharge occurs at a voltage level or not. Censored data are a kind of incomplete data (10, p. 24). In the next two subsections, let us introduce methods dealing with observation values, which are called complete data. As said in Section 1, the recent improvement of voltage measuring instruments has made these methods available.

2.3 New step-up method The new step-up method is a counterpart of the step-up method that deals with complete data. In the test procedures, the difference between them is only whether a voltage, say u_j, at the moment when a disruptive discharge occurs is supposed to be recorded or not.

The likelihood function is given as follows (5)~(8):

$$L = \prod_{j=1}^{n} \left\{ f(u_j;\boldsymbol{\theta})[1-F(U_{\delta_j};\boldsymbol{\theta})]^{m_j-1} \times \prod_{k=1}^{\delta_j-1}[1-F(U_k;\boldsymbol{\theta})]^{m} \right\} \cdots\cdots\cdots (4)$$

All the test results are expressed by $\triangle U$ and $\{(\delta_j, m_j, u_j)\}_{j=1}^{n}$.

2.4 New up-and-down method The new up-and-down method is a counterpart of the up-and-down method that deals with complete data. In the test procedures, the difference between them is the same as the difference between the step-up method and the new step-up method.

The likelihood function is given as follows (9):

$$L = \prod_{i=1}^{N}[f(u_i;\boldsymbol{\theta})]^{d_i}[1-F(U_i;\boldsymbol{\theta})]^{1-d_i}. \cdots\cdots\cdots (5)$$

2.5 Remarks on methods In the papers (5)~(9), it is reported that the new step-up method and the new up-and-down method give more precise estimation than that by the step-up method and the up-and-down method, respectively. This is one of the virtues of including complete data. These methods, however, have a technical problem that is not resolved. A disruptive discharge causes a rapid collapse on an impulse voltage to zero or nearly to zero. The collapse can occur before or after the peak of the impulse voltage. Let us call the former or latter case a disruptive discharge on the front or tail of the impulse, respectively, (1, pp. 119-120). For example, Figure 1 or 2 shows a disruptive discharge on the front or tail of an impulse. These collapses are different phenomenon. Although it is not clearly understood which voltage should be used as a voltage datum for a disruptive discharge when it occurs on the tail of an impulse, some voltage is used as a complete datum in the methods (5) (9). (For example, the peak voltage is used

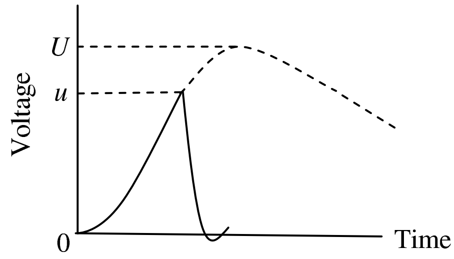

Fig. 1. Disruptive discharge on the front.

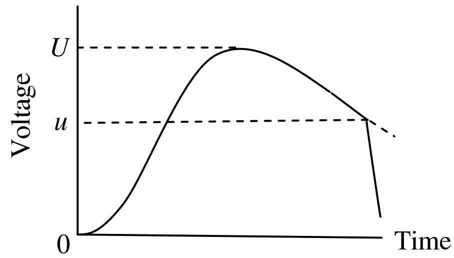

Fig. 2. Disruptive discharge on the tail.

in the paper (5).) This defect will be removed by some modified methods in Section 4.

The up-and-down method has been proposed by Dixon and Mood (11) and they have given the following explanation concerning U_1: if U_1 is poorly chosen, the early observations from U_1 to some U_i will be spent in getting from U_1 to the region of the mean; they will obviously contribute little to the more precise location of the mean. According to this, the step-up method seems to be inferior to the up-and-down method. In the next section we will see that it is not always true.

3. Comparison of methods

We first show the asymptotic errors of parameter estimators in the step-up method and the up-and-down method, and second investigate how many times an experimenter needs to test to obtain good estimates whose errors are close to the asymptotic errors. In the sequel the discharge probability distribution is supposed to be a normal distribution with mean μ and standard deviation σ.

3.1 Asymptotic unit errors We define the asymptotic unit errors of the ML estimators of μ and σ by $[n(\mathcal{I}^{-1})_{11}]^{1/2}$ and $[n(\mathcal{I}^{-1})_{22}]^{1/2}$, where $\mathcal{I}$ stands for the Fisher information matrix. In order to seek the matrix in the step-up method, let us rewrite (1). Denote by λ_i and ν_i the total numbers of discharges and withstands in the voltage applications at a voltage level U_i, respectively. These are expressed by

$$\lambda_i = \sum_{j=1}^{n} I_i(\delta_j),$$

$$\nu_i = \sum_{j=1}^{n}\Big(m\tilde{I}_i(\delta_j) + (m_j-1)I_i(\delta_j)\Big),$$

where $I_i(k) \stackrel{\text{def}}{=} 1$ if $i=k$ or 0 otherwise, and $\tilde{I}_i(k) \stackrel{\text{def}}{=} 1$ if $i<k$ or 0 otherwise. By utilizing these expressions, we can obtain

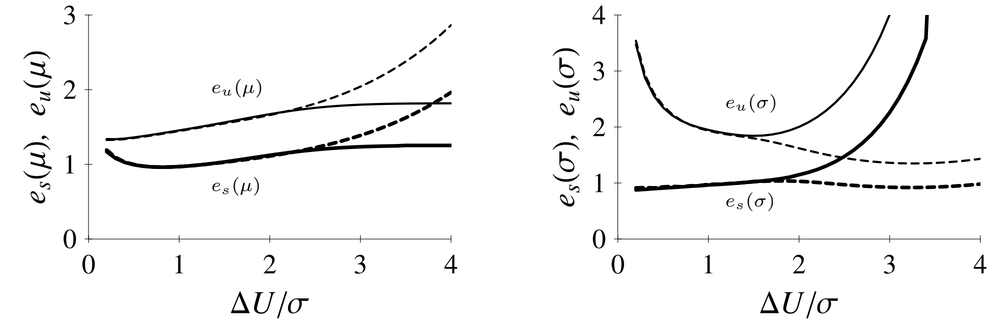

Fig. 3. Asymptotic unit errors of the step-up method or the up-and-down method. (Solid lines: Case A; dotted lines: Case B.)

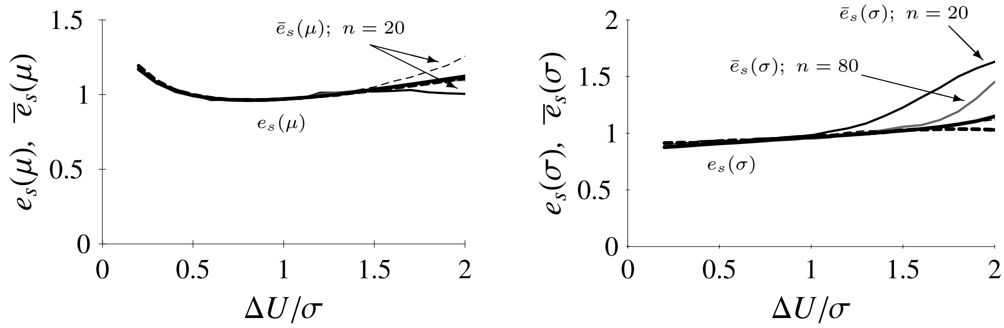

Fig. 4. Empirical or asymptotic unit errors of the step-up method. (Solid lines: Case A; dotted lines: Case B.)

$$L=\prod_{i\geq 1}[F(U_i;\mu,\sigma)]^{\lambda_i}[1-F(U_i;\mu,\sigma)]^{\nu_i}\cdots\cdots(6)$$

from (1). Now, let us seek the Fisher information matrix for (6). Denote $F(U_i;\mu,\sigma)$, $1-p_i$ and $\prod_{k=1}^{m} q_k^m$ by p_i, q_i and r_i, respectively. In addition, introduce the following symbols:

$$x_i \stackrel{\text{def}}{=} (U_i-\mu)/\sigma, \qquad z_i \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x_i^2}{2}\right),$$

$$A_i \stackrel{\text{def}}{=} \begin{bmatrix} 1 & x_i \\ x_i & x_i^2 \end{bmatrix}.$$

From (6) the Fisher information matrix in the step-up method becomes:

$$\mathcal{I}=\frac{n}{\sigma^2}\sum_{i\geq 1} r_{i-1}\left(\sum_{k=0}^{m-1} q_i^k\right)\frac{z_i^2}{p_i q_i}A_i\cdots\cdots\cdots\cdots(7)$$

since the expectations $E[\lambda_i]$ and $E[\nu_i]$ are expressed by

$$\sum_{j=1}^{n}\left(r_{i-1}\sum_{k=0}^{m-1} q_i^k p_i\right)=nr_{i-1}(1-q_i^m)\cdots\cdots\cdots(8)$$

and

$$\sum_{j=1}^{n}\left(mr_i+r_{i-1}\sum_{k=0}^{m-1} kq_i^k p_i\right)=nr_{i-1}\sum_{k=1}^{m} q_i^k,\ \cdot(9)$$

respectively.

From (3) the Fisher information matrix in the up-and-down method becomes:

$$\mathcal{I}=\frac{1}{\sigma^2}\sum_{i=1}^{N}\sum_{k=-i}^{i} P[\bar{I}_i(k)=1]\frac{z_k^2}{p_k q_k}A_k,\ \cdots\cdots\cdots(10)$$

where $\bar{I}_i(k) \stackrel{\text{def}}{=} 1$ if $U_i=U_1+k\triangle U$ or 0 otherwise [12].

In the step-up method, let $e_s(\mu)$ be the asymptotic unit error of the ML estimator of μ and $e_s(\sigma)$ that of σ. Similarly, let us denote by $e_u(\mu)$ and $e_u(\sigma)$ the asymptotic unit errors of the ML estimators in the up-and-down method.

Suppose that $\mu=0$ and $\sigma=1$. We investigate the two cases, Case A: the mean falls on a voltage level and Case B: the mean is midway between two voltage levels.

In the step-up method, U_1 needs to be sufficiently low. When $U_1 \leq \mu-3.5\sigma$ and $m=1$, each graph of $e_s(\mu)$ and $e_s(\sigma)$ in the interval of $\triangle U/\sigma$ from 0.2 to 4 keeps the same shape in each case. For this, we set $m=1$ and $U_1=\max_i(\mu-i\triangle U)$ or $\max_i(\mu-(i+0.5)\triangle U)$ under the condition that $U_1 \leq \mu-3.5\sigma$.

On the other hand, in the up-and-down method, U_1 does not need to be low, but it rather needs to be close to μ. For this, let us suppose $U_1=\max_i(\mu-i\triangle U)$ or $\max_i(\mu-(i+0.5)\triangle U)$ under the condition that $U_1 \leq \mu-\sigma$. Then, each graph of $e_u(\mu)$ and $e_u(\sigma)$ also keeps almost the same shape if $N \geq 40$. This means that the sample size 40 is large enough for us to know how the errors asymptotically behave in the up-and-down method. Thus, we set $N=40$.

We show $e_s(\mu)$, $e_u(\mu)$, $e_s(\sigma)$ and $e_u(\sigma)$ in Figure 3. The thick or normal lines correspond to the step-up method or the up-and-down method, respectively. Throughout the present paper, the solid or dotted lines correspond to Case A or B, respectively. The figure tells us that the step-up method is quite superior to the up-and-down method in the asymptotic unit errors. Especially, it is remarkable that $e_s(\mu)$ and $e_s(\sigma)$ are small, and in the interval of $\triangle U/\sigma$ from 0.2 to 1.8, each of

Table 1. Average number of voltage stress applications.

$\triangle U/\sigma$	0.4	0.8	1.2	1.6	2.0
Case A	9.0	5.1	4.4	3.4	2.5
Case B	7.5	4.6	3.9	2.9	3.0

them is almost the same in Cases A and B.

3.2 Evaluation in small samples We have seen that the step-up method can provide asymptotically good estimations in the interval $[0.2, 1.8]$. This is not, however, sufficient to approve the performance of the method. In this subsection, let us evaluate it in small samples by means of Monte Carlo simulation.

We define the empirical unit error of an ML estimator by $n^{1/2}$ times its root mean square error, and denote the empirical unit errors of μ and σ in the step-up method by $\bar{e}_s(\mu)$ and $\bar{e}_s(\sigma)$. When $n = 20$ or 80, 10000 sets of independent pseudo-random samples were considered for each value of $\triangle U/\sigma (= 0.2, 0.3, \ldots, 2.0)$.

Figure 4 gives $\bar{e}_s(\mu)$ and $\bar{e}_s(\sigma)$ for $n = 20$, 80 as well as $e_s(\mu)$ and $e_s(\sigma)$ for comparison. The thick lines correspond to the asymptotical unit errors, and normal or gray lines correspond to the empirical unit errors for $n = 20$ or 80, respectively. From the figure we can see the following: when $n = 20$, the empirical unit errors go away from the asymptotic unit errors as $\triangle U/\sigma$ becomes close to 2; when $n = 80$, the empirical unit errors are almost the same as the asymptotic unit errors except $\bar{e}_s(\sigma)$ in Case A.

Next, let us seek the average of the number of voltage stress applications necessary to obtain one disruptive discharge. By adding (8) and (9), substituting $n = 1$ into it and taking a summation over possible values of i, we obtain the average: $\sum_{i\geq 1} r_{i-1}(\sum_{k=0}^{m-1} q_i^k)$. Table 1 shows its values when $m = 1$. From the table, for example, we can see that it may be necessary to apply voltage stresses 272 times when $n = 80$ and $\triangle U/\sigma = 1.6$ in Case A. (Note that the number $n = 80$ comes from the last sentence in the previous paragraph.)

Summarizing what we have seen in this section, we can say the following:

- The step-up method is superior to the up-and-down method in the asymptotic unit errors. This is a new finding, which is different from the explanation concerning U_1 in the paper (11). (For details, remember the last paragraph in Subsection 2.4.)
- In order to attain the empirical unit errors as small as the asymptotic unit errors, however, the step-up method requires a much larger number of voltage stress applications than the up-and-down method does. This is also a new finding, which has not been shown in the papers (3) (5).

4. Modified methods

In Section 2, we have seen that the new step-up method and the new up-and-down method contain the technical problem. To avoid it, let us consider new other methods that use a disruptive discharge voltage as a complete datum only if a disruptive discharge occurs on the front of an impulse and that deal with disruptive discharges on the tails in the same way as the up-and-down method does. That is, these methods use a censored datum if a disruptive discharge occurs on the tail. We call them the modified step-up method and the modified up-and-down method.

4.1 Asymptotic unit errors First, we seek the likelihood function and the Fisher information matrix in the modified step-up method. For the jth disruptive discharge ($1 \leq j \leq n$), define

$$\tau_j \stackrel{\text{def}}{=} \begin{cases} 1 & \text{(it occurs on the tail),} \\ 0 & \text{(it occurs on the front).} \end{cases}$$

The likelihood function becomes:

$$L = \prod_{j=1}^{n} \Bigg\{ [F(U_{\delta_j}; \boldsymbol{\theta})]^{\tau_j} [f(u_j; \boldsymbol{\theta})]^{1-\tau_j} \times [1 - F(U_{\delta_j}; \boldsymbol{\theta})]^{m_j - 1} \prod_{k=1}^{\delta_j - 1} [1 - F(U_k; \boldsymbol{\theta})]^m \Bigg\}.$$

As in Subsection 3.1, this can be rewritten into

$$L = \prod_{i\geq 1} \Bigg\{ \prod_{j=1}^{n} \Big[[F(U_{\delta_j}; \mu, \sigma)]^{\tau_j} [f(u_j; \mu, \sigma)]^{1-\tau_j} \Big]^{I_i(\delta_j)} \times [1 - F(U_i; \mu, \sigma)]^{\nu_i} \Bigg\} \cdots\cdots\cdots\cdots (11)$$

For ease of analysis, let us suppose that disruptive discharges occur on the tails independently of voltage stress levels. Then, we can set

$$P[\tau_j = 1 | I_i(\delta_j) = 1] = \gamma,$$

where γ is a constant. By similar calculations to those in Subsection 3.1, from (11) the Fisher information matrix in the modified step-up method is given as follows:

$$\mathcal{I} = \frac{n}{\sigma^2} \sum_{i\geq 1} r_{i-1} \left(\sum_{k=0}^{m-1} q_i^k \right) C_i, \cdots\cdots\cdots\cdots\cdots (12)$$

where

$$B_i \stackrel{\text{def}}{=} \begin{bmatrix} -x_i z_i + p_i & -z_i - x_i^2 z_i \\ -z_i - x_i^2 z_i & -x_i z_i - x_i^3 z_i + 2p_i \end{bmatrix},$$

$$C_i \stackrel{\text{def}}{=} \left(\frac{(p_i + \gamma q_i) z_i^2}{p_i q_i} A_i + (1 - \gamma) B_i \right).$$

On the other hand, by similar calculations to those in the up-and-down method and the modified step-up method, the Fisher information matrix in the modified up-and-down method becomes:

$$\mathcal{I} = \frac{1}{\sigma^2} \sum_{i=1}^{N} \sum_{k=-i}^{i} P[\bar{I}_i(k) = 1] C_k. \cdots\cdots\cdots\cdots (13)$$

The following are remarkable:

- The expressions in the right-hand side of (7) and (12) or (10) and (13) are the same except the difference between $z_i^2 A_i / p_i q_i$ and C_i or $z_k^2 A_k / p_k q_k$ and C_k.

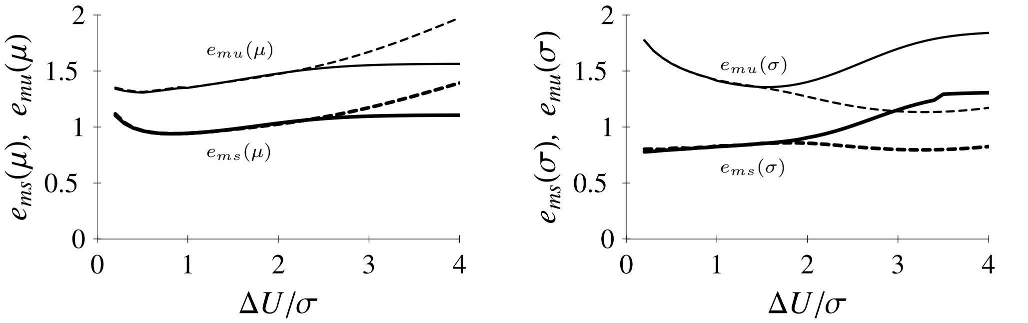

Fig. 5. Asymptotic unit errors of the modified step-up method or the modified up-and-down method when $\gamma = 0.7$.
(Solid lines: Case A; dotted lines: Case B.)

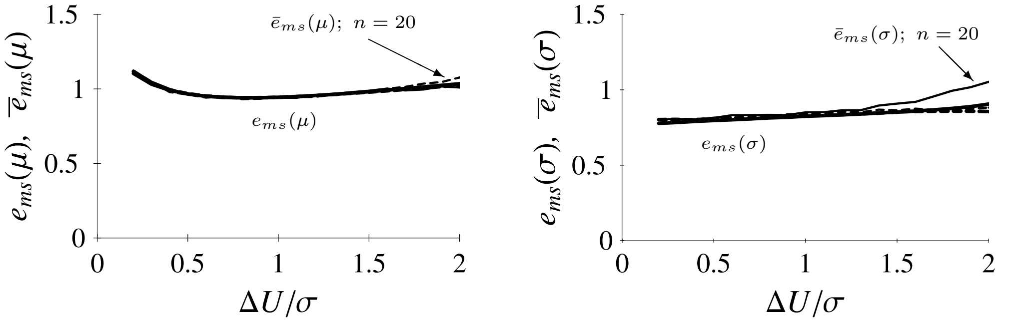

Fig. 6. Empirical or asymptotic unit errors of the modified step-up method when $\gamma = 0.7$.
(Solid lines: Case A; dotted lines: Case B.)

- When $\gamma = 1$, (12) is equivalent to (7), whereas (13) is equivalent to (10).

We denote the asymptotic unit errors of the ML estimators in the modified step-up method or the modified up-and-down method by $e_{ms}(\mu)$ and $e_{ms}(\sigma)$ or $e_{mu}(\mu)$ and $e_{mu}(\sigma)$.

In addition to the setting of μ, σ, m, U_1 and N in Subsection 3.1, we set $\gamma = 0.7$. Then, Figure 5 gives $e_{ms}(\mu)$, $e_{mu}(\mu)$, $e_{ms}(\sigma)$ and $e_{mu}(\sigma)$. The thick or normal lines correspond to the modified step-up method or the modified up-and-down method, respectively. The figure shows that the modified step-up method is superior to the modified up-and-down method in the asymptotic unit errors, and in the interval of $\triangle U/\sigma$ from 0.2 to 2.0, each of $e_{ms}(\mu)$ and $e_{ms}(\sigma)$ is almost the same in Cases A and B.

By comparing Figures 3 and 5, we can see that the errors in Figure 5 are smaller as a whole. This is because more precise estimation can be given by using complete data even only for disruptive discharges on the fronts of impulses.

4.2 Evaluation in small samples We evaluate the performance of the modified step-up method in small samples by means of Monte Carlo simulation. We denote by $\bar{e}_{ms}(\mu)$ and $\bar{e}_{ms}(\sigma)$ the empirical unit errors of the ML estimators of μ and σ in the modified step-up method. Under the setting that $n = 20$ and $\gamma = 0.7$, 10000 sets of independent pseudo-random samples were considered for each simulation. Figure 6 shows $\bar{e}_{ms}(\mu)$ and $\bar{e}_{ms}(\sigma)$ by normal lines as well as $e_{ms}(\mu)$ and $e_{ms}(\sigma)$ by thick lines for comparison. From this we can see that the empirical unit errors are very close to the asymptotic unit errors throughout the interval $[0.2, 1.4]$. Since $n = 20$, for example, Table 1 tells us that it is necessary to apply voltage stresses about 80 or 90 times when $\triangle U/\sigma = 1.2$.

Compared with the step-up method, the modified step-up method can reduce the number of voltage stress applications necessary to attain such small empirical unit errors. This is one of the virtues of including complete data.

4.3 Concrete example We give an example to show the difference between the up-and-down method and the modified up-and-down method. The modified up-and-down method is the same as the up-and-down method in the test procedure except the following: when a disruptive discharge occurs on the front of an impulse, the voltage at the moment when it occurs needs to be recorded in the modified up-and-down method. Let us give such data in Table 2, which have been simulated under the setting that $\mu = 40$, $\sigma = 1.6$ and $\gamma = 0.7$. The table shows that disruptive discharges occurred on the front of the impulse when $i = 4$, 10, 13, 14, 31, 33 and 40. Each voltage u_i at the moment when a disruptive discharge occurred on the front is given in the parenthesis.

Noting that u_i's are utilized to seek the estimates of μ and σ in the modified up-and-down method, we obtain $\hat{\mu} = 39.7$ and $\hat{\sigma} = 2.09$ as the ML estimates by means of the formula in Appendix, and then obtain 0.44 and 0.48 as the estimates of the standard errors of them, respectively, by $[(\mathcal{I}_{ob}^{-1})_{11}]^{1/2}$ and $[(\mathcal{I}_{ob}^{-1})_{22}]^{1/2}$ in Appendix. On the other hand, noting that u_i's are not utilized in the up-and-down method, we obtain $\hat{\mu} = 39.5$, $\hat{\sigma} = 2.57$ and the estimates of the standard errors 0.56 and 0.87 in a similar way. Here, it should be remembered that the errors of parameter estimates in the modified up-and-down method are less influenced by an unknown value $\triangle U/\sigma$ than the up-and-down method. (See Figures 3 and 5.)

5. Conclusions

First, we have stated that, from the viewpoint of statistical independence, the step-up method is a method for self-restoring insulation. Second, we have investigated the performance of the step-up, the up-and-down, the modified step-up and the modified up-and-down methods as a method for self-restoring insulation. Our conclusions are as follows.

- The asymptotical unit errors of the ML estimators in the step-up method or the modified step-up method are smaller than those in the up-and-down method or the modified up-and-down method, respectively. This is a merit which is obtained by using the step-up procedure under the statistical independence of all voltage stress applications.
- In order to attain the empirical unit errors as small as them, however, the step-up method demands a much larger number of voltage stress applications than 40, which is sufficient in the up-and-down method. This disadvantage is reduced in the modified step-up method. This is one of the virtues of including complete data. Even the method, however, still demands a larger number of voltage stress applications than 40. That is, only in the case that the number is allowed to be about 100, it is effective.
- The asymptotical unit errors of the ML estimators in the modified step-up method or the modified up-and-down method are less influenced by an unknown value $\triangle U/\sigma$ than those in the step-up method or the up-and-down method. This is also one of the virtues of including complete data.
- As shown in the papers (5) (9) (12), the asymptotical unit errors in the new step-up method or the new up-and-down method are also not influenced so much by $\triangle U/\sigma$. The methods, however, have the technical problem. That is, although it is not clearly understood which voltage should be used as a voltage datum for a disruptive discharge when it occurs on the tail of an impulse, some voltage is used as a complete datum in the methods. The modified step-up method or the modified up-and-down method removes this defect.
- Whereas the step-up procedure requires choosing a sufficiently low voltage level for every object as U_1, the up-and-down procedure allows us to choose it roughly. In addition, the modified up-and-down method makes it possible to choose $\triangle U$ roughly since the errors of the ML estimators are less sensitive to its value than the up-and-down method. Consequently, we can roughly choose U_1 and $\triangle U$ in the modified up-and-down method.

Table 2. Data example.

i	d_i	U_i (u_i)	i	d_i	U_i (u_i)	i	d_i	U_i (u_i)	i	d_i	U_i (u_i)
1	1	38.0	11	0	38.0	21	1	38.0	32	0	40.0
2	0	36.0	12	0	40.0	22	0	36.0	33	1	42.0
3	0	38.0	13	1	42.0	23	0	38.0			(40.8)
4	1	40.0			(41.7)	24	1	40.0	34	1	40.0
		(37.1)	14	1	40.0	25	1	38.0	35	0	38.0
5	0	38.0			(39.0)	26	0	36.0	36	1	40.0
6	1	40.0	15	0	38.0	27	0	38.0	37	0	38.0
7	1	38.0	16	1	40.0	28	0	40.0	38	0	40.0
8	0	36.0	17	0	38.0	29	0	42.0	39	0	42.0
9	0	38.0	18	1	40.0	30	1	44.0	40	1	44.0
10	1	40.0	19	0	38.0	31	1	42.0			(38.5)
		(39.6)	20	1	40.0			(39.8)			

Appendix

By utilizing the expectation-maximization algorithm [10] as in the paper (12), we give an iterative formula to get the ML estimates and a formula to calculate an observed information matrix. Let n_t, n and w denote the number of disruptive discharges on the tails of impulses, the number of disruptive discharges and the number of withstands, respectively. Of all the voltage levels $\{U_i\}_{i=1}^{w+n}$, pick up the voltage levels $\{U_{(i)}\}_{i=1}^{w}$ for withstands and $\{U_{(i)}\}_{i=w+1}^{w+n_t}$ for discharges on the tails. On the other hand, of all the discharge voltages $\{u_i\}_{i=1}^{n}$, pick up the discharge voltages on the fronts $\{u_{(i)}\}_{i=1}^{n-n_t}$. Before we see formulas, we should note that each of them has the same in both of the modified step-up method and the modified up-and-down method. That is, whereas the formulas give results in the modified step-up method if data in the modified step-up method are given, they give results in the modified up-and-down method if data in the modified up-and-down method are given.

The iterative formula for the ML estimation is

$$\mu^{(k+1)} = \frac{w+n_t}{w+n}\mu^{(k)} + \frac{1}{w+n}\sum_{i=1}^{n-n_t} u_{(i)} + \frac{1}{w+n}\left(\sum_{i=1}^{w} D_i^{(k)} + \sum_{i=w+1}^{w+n_t} E_i^{(k)}\right),$$

$$\sigma^{(k+1)} = \Bigg\{ \frac{w+n_t}{w+n}\left[\left(\sigma^{(k)}\right)^2 + \left(\triangle\mu_1^{(k)}\right)^2\right] + \frac{1}{w+n}\sum_{i=1}^{n-n_t}\left(u_{(i)} - \mu^{(k+1)}\right)^2 + \frac{1}{w+n}\left[\sum_{i=1}^{w}\left(U_{(i)} + \triangle\mu_2^{(k)}\right)D_i^{(k)} + \sum_{i=w+1}^{w+n_t}\left(U_{(i)} + \triangle\mu_2^{(k)}\right)E_i^{(k)}\right] \Bigg\}^{1/2},$$

where

$$D_i^{(k)} \stackrel{\text{def}}{=} \left(\sigma^{(k)}\right)^2 \frac{f(U_{(i)}; \mu^{(k)}, \sigma^{(k)})}{1 - F(U_{(i)}; \mu^{(k)}, \sigma^{(k)})},$$

$$E_i^{(k)} \stackrel{\text{def}}{=} -\left(\sigma^{(k)}\right)^2 \frac{f(U_{(i)}; \mu^{(k)}, \sigma^{(k)})}{F(U_{(i)}; \mu^{(k)}, \sigma^{(k)})},$$

$$\triangle\mu_1^{(k)} \overset{\text{def}}{=} \mu^{(k)} - \mu^{(k+1)}, \quad \triangle\mu_2^{(k)} \overset{\text{def}}{=} \mu^{(k)} - 2\mu^{(k+1)}.$$

When $\mu^{(0)}$ and $\sigma^{(0)}$ are properly given, the formula provides two series of approximates $\{\mu^{(k)}\}_{k\geq 1}$ and $\{\sigma^{(k)}\}_{k\geq 1}$ to the ML estimates of μ and σ. One of the good selections for $\mu^{(0)}$ and $\sigma^{(0)}$ is a pair of the sample mean and the sample standard deviation of $\{u_{(i)}\}_{i=1}^{n-n_t} \cup \{U_{(i)}\}_{i=w+1}^{w+n_t}$.

Each element of the observed information matrix $\mathcal{I}_{ob}$ is given as follows:

$$(\mathcal{I}_{ob})_{11} = \frac{1}{\hat{\sigma}^4}\left[\sum_{i=1}^{n-n_t} \bar{u}_{(i)}^2 + \sum_{i=1}^{w} D_i^2 + \sum_{i=w+1}^{w+n_t} E_i^2\right],$$

$$(\mathcal{I}_{ob})_{12} = -\frac{1}{\hat{\sigma}^5}\sum_{i=1}^{w} D_i\left[\bar{U}_{(i)}(\bar{U}_{(i)} - D_i) + \hat{\sigma}^2\right] - \frac{1}{\hat{\sigma}^5}\sum_{i=w+1}^{w+n_t} E_i\left[\bar{U}_{(i)}(\bar{U}_{(i)} - E_i) + \hat{\sigma}^2\right],$$

$$(\mathcal{I}_{ob})_{22} = \frac{2(n-n_t)}{\hat{\sigma}^2} - \frac{1}{\hat{\sigma}^6}\sum_{i=1}^{w} \bar{U}_{(i)} D_i\left[\bar{U}_{(i)}^2 - \bar{U}_{(i)} D_i + \hat{\sigma}^2\right] - \frac{1}{\hat{\sigma}^6}\sum_{i=w+1}^{w+n_t} \bar{U}_{(i)} E_i\left[\bar{U}_{(i)}^2 - \bar{U}_{(i)} E_i + \hat{\sigma}^2\right],$$

where $(\mathcal{I}_{ob})_{21} = (\mathcal{I}_{ob})_{12}$,

$$\bar{u}_{(i)} \overset{\text{def}}{=} u_{(i)} - \hat{\mu}, \quad D_i \overset{\text{def}}{=} \hat{\sigma}^2 \frac{f(U_{(i)};\hat{\mu},\hat{\sigma})}{1 - F(U_{(i)};\hat{\mu},\hat{\sigma})},$$

$$E_i \overset{\text{def}}{=} -\hat{\sigma}^2 \frac{f(U_{(i)};\hat{\mu},\hat{\sigma})}{F(U_{(i)};\hat{\mu},\hat{\sigma})}, \quad \bar{U}_{(i)} \overset{\text{def}}{=} U_{(i)} - \hat{\mu}.$$

Lastly, it is also remarkable that these formulas give results in the step-up method or the up-and-down method if $n = n_t$, whereas they give results in the new step-up method or the new up-and-down method if $n_t = 0$.

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